Lecture: Introduction to LP, SDP and SOCP

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Linear Programming (LP)

Primal		**Dual**	
min	$c_1x_1 + \ldots + c_nx_n$	max	$b_1y_1 + \ldots + b_my_m$
s.t.	$a_{11}x_1 + \ldots + a_{1n}x_n = b_1$	s.t.	$a_{11}y_1 + \ldots + a_{m1}y_m \leq c_1$
	$\ldots$		$\ldots$
	$a_{m1}x_1 + \ldots + a_{mn}x_n = b_m$		$a_{1n}y_1 + \ldots + a_{mn}y_m \leq c_n$
	$x_i \geq 0$		

more succinctly

Primal (P)

$$\begin{aligned} \min \quad & c^\top x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

Dual (D)

$$\begin{aligned} \max \quad & b^\top y \\ \text{s.t.} \quad & A^\top y + s = c \\ & s \geq 0 \end{aligned}$$

Weak duality

Suppose

- x is feasible to (P)
- (y, s) is feasible to (D)

Then

$$\begin{aligned} 0 &\leq x^\top s \quad \text{because } x_i s_i \geq 0 \\ &= x^\top (c - A^\top y) \\ &= c^\top x - (Ax)^\top y \\ &= c^\top x - b^\top y \\ &= \text{duality gap} \end{aligned}$$

Key Properties of LP

- Strong duality: If both Primal and Dual are feasible then at the optimum
$$c^\top x = b^\top y \iff x^\top s = 0$$
- complementary slackness: This implies
$$\begin{aligned} x^\top s &= x_1 s_1 + \ldots + x_n s_n = 0 \quad \text{and therefore} \\ x_i s_i &= 0 \end{aligned}$$

complementarity

- Putting together primal feasibility, dual feasibility and complementarity together we get a square system of equations

$$\begin{aligned} Ax &= b \\ A^\top y + s &= c \\ x_i s_i &= 0 \quad \text{for } i = 1, \ldots, n \end{aligned}$$

- At least in principle this system determines the primal and dual optimal values

Algebraic characterization

- We can define $x \circ s = (x_1 s_1, \ldots, x_n s_n)^\top$ and

$$\mathsf{L}_x : y \to (x_1 y_1, \ldots, x_n y_n)^\top \text{ i.e. } \mathsf{L}_x = \mathrm{Diag}(x)$$

- We can write complementary slackness conditions as

$$x \circ s = \mathsf{L}_x s = \mathsf{L}_x \mathsf{L}_s 1 = 0$$

- 1, the vector of all ones, is the identity element:

$$x \circ 1 = x$$

Semidefinite Programming (SDP)

- $X \succeq Y$ means that the the symmetric matrix $X - Y$ is positive semidefinite
- X is positive semidefinite

$$a^\top Xa \geq 0 \text{ for all vector } a \iff X = B^\top B \iff$$

$$\text{all eigenvalues of } X \text{ is nonnegative}$$

Semidefinite Programming (SDP)

- $\langle X, Y\rangle = \sum_{ij} X_{ij} Y_{ij} = \mathrm{Tr}(XY)$

Primal (P)

$$\begin{array}{ll} \min & \langle C_1, X_1\rangle + \ldots + \langle C_n, X_n\rangle \\ \text{s.t.} & \langle A_{11}, X_1\rangle + \ldots + \langle A_{1n}, X_n\rangle = b_1 \\ & \ldots \\ & \langle A_{m1}, X_1\rangle + \ldots + \langle A_{mn}, X_n\rangle = b_m \\ & X_i \succeq 0 \end{array}$$

Dual (D)

$$\begin{array}{ll} \max & b_1 y_1 + \ldots + b_m y_m \\ \text{s.t.} & A_{11} y_1 + \ldots + A_{m1} y_m + S_1 = c_1 \\ & \ldots \\ & A_{1n} y_1 + \ldots + A_{mn} y_m + S_n = c_n \\ & S_i \succeq 0 \end{array}$$

Simplified SDP

- For simplicity we deal with single variable SDP:

Primal (P)

$$\begin{aligned} \min \quad & \langle C, X \rangle \\ \text{s.t.} \quad & \langle A_1, X \rangle = b_1 \\ & \dots \\ & \langle A_m, X \rangle = b_m \\ & X \succeq 0 \end{aligned}$$

Dual (D)

$$\begin{aligned} \max \quad & b^\top y \\ \text{s.t.} \quad & \sum_i y_i A_i + S = C \\ & S \succeq 0 \end{aligned}$$

- A single variable LP is trivial
- But a single matrix SDP is as general as a multiple matrix

Weak duality in SDP

- Just as in LP
$$\langle X, S\rangle = \langle C, X\rangle - b^\top y$$
- Also if both $X \succeq 0$ and $S \succeq 0$ then
$$\langle X, S\rangle = \mathrm{Tr}(XS^{1/2}S^{1/2}) = \mathrm{Tr}(S^{1/2}XS^{1/2}) \geq 0$$
because $S^{1/2}XS^{1/2} \succeq 0$
- Thus
$$\langle X, S\rangle = \langle C, X\rangle - b^\top y \geq 0$$

Complementarity Slackness Theorem

- $X \succeq 0$ and $S \succeq 0$ and $\langle X, S \rangle = 0$ implies
$$XS = 0$$
- Proof:
$$\langle X, S \rangle = \mathrm{Tr}(XS^{1/2}S^{1/2}) = \mathrm{Tr}(S^{1/2}XS^{1/2})$$
Thus $\mathrm{Tr}(S^{1/2}XS^{1/2}) = 0$. Since $S^{1/2}XS^{1/2} \succeq 0$, then
$$S^{1/2}XS^{1/2} = 0 \Longrightarrow S^{1/2}X^{1/2}X^{1/2}S^{1/2} = 0$$
$$X^{1/2}S^{1/2} = 0 \Longrightarrow XS = 0$$

Algebraic properties of SDP

- For reasons to become clear later it is better to write complementary slackness conditions as

$$\frac{XS + SX}{2} = 0$$

- It can be shown that if $X \succeq 0$ and $S \succeq 0$, then $XS = 0$ iff

$$XS + SX = 0$$

Algebraic properties of SDP

- Definition: $X \circ S = \frac{XS+SX}{2}$
- The binary operation $\circ$ is commutative $X \circ S = S \circ X$
- $\circ$ is not associative: $X \circ (Y \circ Z) \neq (X \circ Y) \circ Z$ in general
- But $X \circ (X \circ X) = (X \circ X) \circ X$. Thus $X^{\circ p} = X^p$ is well defined
- In general $X \circ (X^2 \circ Y) = X^2 \circ (X \circ Y)$
- The identity matrix I is identity w.r.t $\circ$
- Define the operator

$$\mathsf{L}_X : Y \to X \circ Y, \text{ thus } X \circ S = \mathsf{L}_X(S) = \mathsf{L}_X(\mathsf{L}_S(I))$$

Constraint Qualifications

- Unlike LP we need some conditions for the optimal values of Primal and Dual SDP to coincide
- Here are two:
  - If there is primal-feasible $X \succ 0$ (i.e. X is positive definite)
  - If there is dual-feasible $S \succ 0$
- When strong duality holds $\langle X, S \rangle = 0$

KKT Condition

- Thus just like LP The system of equations

$$
\begin{aligned}
\langle A_i, X \rangle &= b_i, \quad \text{for } i = 1, \ldots, m \\
\sum_i y_i A_i + S &= C \\
X \circ S &= 0
\end{aligned}
$$

Gives us a square system

Second Order Cone Programming (SOCP)

- For simplicity we deal with single variable SOCP:

Primal (P)

$$\begin{aligned} \min \quad & c^\top x \\ \text{s.t.} \quad & Ax = b \\ & x_{\mathcal{Q}} \succeq 0 \end{aligned}$$

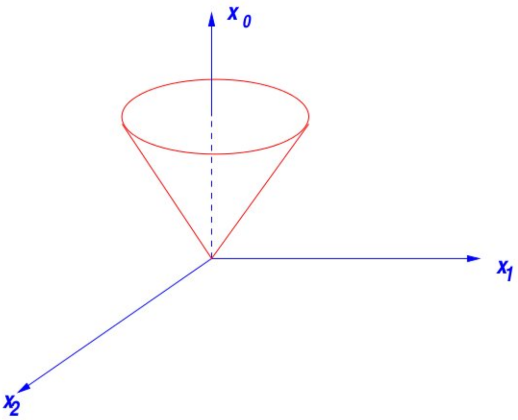
Dual (D)

$$\begin{aligned} \max \quad & b^\top y \\ \text{s.t.} \quad & A^\top y + s = c \\ & s_{\mathcal{Q}} \succeq 0 \end{aligned}$$

- the vectors x, s, c are indexed from zero
- If $z = (z_0, z_1, \ldots, z_n)^\top$ and $\bar{z} = (z_1, \ldots, z_n)^\top$

$$z_{\mathcal{Q}} \geq 0 \iff z_0 \geq \|\bar{z}\|$$

Illustration of SOC



$$\mathcal{Q} = \{z \mid z_0 \geq \|\bar{z}\|\}$$

Weak Duality in SOCP

- The single block SOCP is not as trivial as LP but it still can be solved analytically
- weak duality: Again as in LP and SDP

$$x^\top s = c^\top x - b^\top y = \text{ duality gap}$$

If $x, s \succeq_Q 0$, then

$$\begin{aligned} x^\top s &= x_0 s_0 + \bar{x}^\top \bar{s} \geq \\ &\geq \|\bar{x}\| \cdot \|\bar{s}\| + \bar{x}^\top \bar{s} \quad \text{since } x, s \succeq_Q 0 \\ &\geq |\bar{x}^\top \bar{s}| + \bar{x}^\top \bar{s} \quad \text{Cauchy-Schwartz inequality} \\ &\geq 0 \end{aligned}$$

Complementary Slackness for SOCP

- Given $x \succeq_{\mathcal{Q}} 0$, $s \succeq_{\mathcal{Q}} 0$ and $x^\top s = 0$. Assume $x_0 > 0$ and $s_0 > 0$
- We have

$$(*) \quad x_0^2 \geq \sum_{i=1}^n x_i^2$$

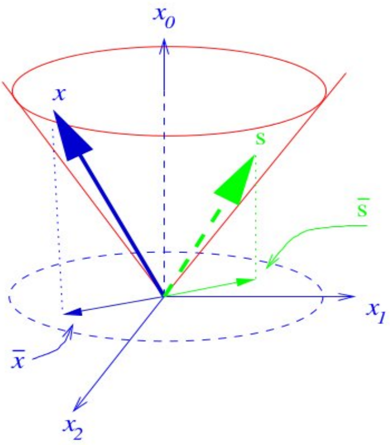
$$(**) \quad s_0^2 \geq \sum_{i=1}^n s_i^2 \Longleftrightarrow x_0^2 \geq \sum_{i=1}^n \frac{s_i^2 x_0^2}{s_0^2}$$

$$(***) \quad x^\top s = 0 \Longleftrightarrow -x_0 s_0 = \sum_i x_i s_i \Longleftrightarrow -2x_0^2 = \sum_{i=1}^n \frac{2x_i s_i x_0}{s_0}$$

- Adding (*), (**), (***), we get $0 \geq \sum_{i=1}^n \left(x_i + \frac{s_i x_0}{s_0}\right)^2$
- This implies

$$x_i s_0 + x_0 s_i = 0, \text{ for } i = 1, \ldots, n$$

Illustration of SOC



When $x \succeq_{\mathcal{Q}} 0$, $s \succeq_{\mathcal{Q}} 0$ are orthogonal both must be on the boundary in such a way that their projection on the $x_1, \ldots, x_n$ plane is collinear

Strong Duality

- at the optimum
$$c^\top x = b^\top y \iff x^\top s = 0$$
- Like SDP constraint qualifications are required
- If there is primal-feasible $x \succ_{\mathcal{Q}} 0$
- If there is dual-feasible $s \succ_{\mathcal{Q}} 0$

Complementary Slackness for SOCP

- Thus again we have a square system

$$\begin{aligned} Ax &= b, \\ A^\top y + s &= c \\ x^\top s &= 0 \\ x_0 s_i + s_0 x_i &= 0 \end{aligned}$$

Algebraic properties of SOCP

- Let us define a binary operation for vectors x and s both indexed from zero

$$\begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix} \circ \begin{pmatrix} s_0 \\ s_1 \\ \vdots \\ s_n \end{pmatrix} = \begin{pmatrix} x^\top s \\ x_0 s_1 + s_0 x_1 \\ \vdots \\ x_0 s_n + s_0 x_n \end{pmatrix}$$

Algebraic properties of SOCP

- The binary operation $\circ$ is commutative $x \circ s = s \circ x$
- $\circ$ is not associative: $x \circ (y \circ z) \neq (x \circ y) \circ z$ in general
- But $x \circ (x \circ x) = (x \circ x) \circ x$. Thus $x^{\circ p} = x^p$ is well defined
- In general $x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$
- The identity matrix I is identity w.r.t $\circ$
- $e = (1, 0, \ldots, 0)^\top$ is the identity: $x \circ e = x$

Algebraic properties of SOCP

- Define the operator

$$\mathsf{L}_x : y \to x \circ y$$

$$\mathsf{L}_x = \mathrm{Arw}(x) = \begin{pmatrix} x_0 & \bar{x}^\top \\ \bar{x} & x_0 I \end{pmatrix}$$

$$x \circ s = \mathrm{Arw}(x)s = \mathrm{Arw}(x)\mathrm{Arw}(s)e$$

Summary

Properties

	LP	SDP	SOCP
binary operator	$x \circ s = (x_i s_i)$	$X \circ S = \frac{XS+SX}{2}$	$x \circ s = \begin{pmatrix} x^\top s \\ x_0 \bar{s} + s_0 \bar{x} \end{pmatrix}$
identity	1	I	$e = (1, 0, \ldots, 0)^\top$
associative	yes	no	no
L_X	$y \to \mathrm{Diag}(x) y$	$Y \to \frac{XY+YX}{2}$	$y \to \mathrm{Arw}(x) y$
Primal feasibility	$Ax = b$	$\langle A_i, X \rangle = b_i$	$Ax = b$
dual feasility	$A^\top y + s = c$	$\sum_i y_i A_i + S = C$	$A^\top y + s = c$
complementarity	$\mathsf{L}_x \mathsf{L}_s 1 = 0$	$\mathsf{L}_X(\mathsf{L}_S(I)) = 0$	$\mathsf{L}_x \mathsf{L}_s e = 0$

Conic LP

A set $\mathsf{K} \subseteq \mathbb{R}^n$ is a proper cone if

- It is a cone: If $x \in \mathsf{K} \implies ax \in \mathsf{K}$ for all $\alpha \geq 0$
- It is convex: $x, y \in \mathsf{K} \implies \alpha x + (1-\alpha)y \in \mathsf{K}$ for $\alpha \in [0, 1]$
- It is pointed: $\mathsf{K} \cap (-\mathsf{K}) = \{0\}$
- It is closed
- It has non-empty interior in $\mathbb{R}^n$
- dual cone:

$$\mathsf{K}^* = \{x \mid \text{ for all } z \in \mathsf{K}, \langle x, z \rangle \geq 0\}$$

Conic LP

Conic-LP is defined as the following optimization problem:

Primal (P)

$$\begin{aligned} \min \quad & c^\top x \\ \text{s.t.} \quad & Ax = b \\ & x \in \mathsf{K} \end{aligned}$$

Dual (D)

$$\begin{aligned} \max \quad & b^\top y \\ \text{s.t.} \quad & A^\top y + s = c \\ & s \in \mathsf{K}^* \end{aligned}$$

- For LP K is the nonnegative orthant
- For SDP K is the cone of positive semidefinite matrices
- For SOCP K is the circular or Lorentz cone
- In all three cases the cones are self-dual $\mathsf{K} = \mathsf{K}^*$